Inhomogeneous Quarter-Wave Transformers of Two Sections*

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Summary—**An inhomogeneous transformer is defined as one in which the guide wavelength is a function of position; for a homogeneous transformer, the guide wavelength is independent of position.**

A previous paper[1] has dealt with inhomogeneous transformers of one section; the existence of an optimum design (which is never homogeneous) was demonstrated. The mathematical tools for inhomogeneous transformers of two or more sections have been presented in another paper.[2] Our purpose here is to apply these results to the solution of the two-section inhomogeneous transformer.

The maximally flat ideal transformer was solved exactly and the design equations verified by subsequent numerical analysis. An approximate procedure to improve the performance over a finite bandwidth (similar to the Tchebycheff response of homogeneous transformers) is also explained.

Introduction

THE NEED for inhomogeneous transformers may arise when it is desired to connect two waveguides of different dimensions. Consider, for instance, rectangular waveguide of dimensions $a \times b$ (Fig. 1). The guide wavelength depends only on the width a. If two waveguides differing only in their b dimensions are to be connected, then a homogeneous quarter-wave transformer is possible (but not necessary and probably not optimum[1]). It has been shown on a first-order theory that where a homogeneous transformer is possible, the shortest maximally flat transformer is a quarter-wave transformer.[3] Since a single-section homogeneous quarter-wave transformer can always be improved by an inhomogeneous design,[1] it seems likely that an inhomogeneous quarter-wave transformer of more than one section will likewise improve the best electrical performance when a maximum transformer length is specified. This result, however, remains to be confirmed.

If two waveguides differing in the a dimension (Fig. 1), as well as possibly the b dimension, are to be connected, then a homogeneous transformer is not possible at all, and again it may be suspected that an inhomogeneous quarter-wave transformer will yield the best performance when the transformer is to be kept below a certain length. Such a discussion then provides the motivation for the following investigation.

Mathematical Summary

In this section, some definitions and formulas will be summarized which have been proved elsewhere,[1,2] and which are needed to derive the design equations of a two-section quarter-wave transformer. Such a transformer is shown in Fig. 2. The two sections are generally of unequal electrical lengths θ_1 and θ_2, which become equal only at center frequency, when

$$\theta_1 = \theta_2 = \frac{\pi}{2} \cdot \tag{1}$$

The four characteristic impedances from input to output are denoted by Z_0, Z_1, Z_2, and Z_3. The log ratios of the three steps are defined[2] by

$$\alpha_i = \frac{1}{2} \ln \frac{Z_i}{Z_{i-1}} \qquad (i = 1, 2, 3) . \tag{2}$$

The spin matrix exponentials,[2] which will be needed to express transformation matrices, are

$$\begin{aligned} E_1(x) &= \exp(\sigma_1 x) \\ E_2(x) &= \exp(\sigma_2 x) \\ E_3(x) &= \exp(\sigma_3 x) \end{aligned} \tag{3}$$

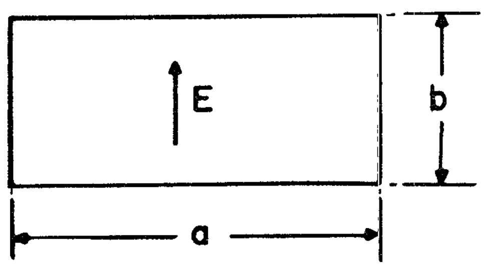

Fig. 1—Waveguide cross section.

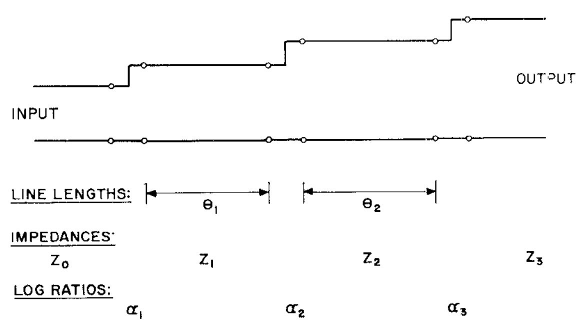

Fig. 2—Two section transformer parameters.

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[1] L. Young, "Optimum quarter-wave transformers," 1960 IRE INTERNATIONAL CONVENTION RECORD, pt. 3, pp. 123–129. Also, IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-8, pp. 478–483; September, 1960.

[2] L. Young, "Spin matrix exponentials and transmission matrices," to be published in *Quart. Appl. Math.*

[3] H. J. Riblet, "A general theorem on an optimum stepped impedance transformer," IRE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-8, pp. 169–170; March, 1960.

where σ_1, σ_2, and σ_3 are the three Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = j\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The transfer matrix,[2,4] of each impedance step is then the first spin matrix exponential of the log ratio of that impedance step,

$$E_1(\alpha), \tag{5}$$

and the transfer matrix of each section of line is given by the third spin matrix exponential of that line length times $\sqrt{-1}$,

$$E_3(j\theta). \tag{6}$$

The over-all transfer matrix of the two-section transformer is given in terms of the two line lengths θ_1, θ_2, and the three log ratios α_1, α_2 and α_3, by the product

$$\boldsymbol{T} = E_1(\alpha_1)E_3(j\theta_1)E_1(\alpha_2)E_3(j\theta_2)E_1(\alpha_3). \tag{7}$$

At center frequency

$$E_3(j\theta_i) = E_3\left(j\frac{\pi}{2}\right) = \sigma_3, \tag{8}$$

and in that case (7) reduces to

$$\boldsymbol{T}_{(\theta_i=\pi/2)} = E_1(\alpha_1 - \alpha_2 + \alpha_3) \tag{9}$$

by means of (8) of Young.[2]

Writing

$$\boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{10}$$

the diagonal part[2] of $\boldsymbol{T}$ is then denoted by

$$\mathrm{Di}(\boldsymbol{T}) = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix} \tag{11}$$

and the antidiagonal part[2] of $\boldsymbol{T}$ by

$$\mathrm{Ag}(\boldsymbol{T}) = \begin{pmatrix} 0 & T_{12} \\ T_{21} & 0 \end{pmatrix}. \tag{12}$$

The condition for a match is

$$\mathrm{Ag}(\boldsymbol{T}) = 0. \tag{13}$$

Therefore, for a match at center frequency, (9) yields

$$\alpha_1 - \alpha_2 + \alpha_3 = 0. \tag{14}$$

Denote the free space (or medium) wavelength by λ and define the differential operator[1]

$$D = \lambda \frac{d}{d\lambda}\cdot \tag{15}$$

The condition for perfect match, as well as zero slope of the frequency response curve at center frequency, is given by (13) combined with

$$\mathrm{Ag}(D\boldsymbol{T}) = 0. \tag{16}$$

From (7) and (8), with (5), (7) and (8) of Young,[2]

$$\begin{aligned} D\boldsymbol{T} = {} & \sigma_1\boldsymbol{T}D\alpha_1 - \sigma_1\boldsymbol{T}D\alpha_2 + \sigma_1\boldsymbol{T}D\alpha_3 \\ & + j\sigma_3E_1(-\alpha_1)\sigma_3E_1(\alpha_2)\sigma_3E_1(\alpha_3)D\theta_1 \\ & + j\sigma_3E_1(-\alpha_1)\sigma_3E_1(-\alpha_2)\sigma_3E_1(\alpha_3)D\theta_2 \end{aligned} \tag{17}$$

$$\begin{aligned} \therefore\ D\boldsymbol{T} = {} & \sigma_1\boldsymbol{T}D(\alpha_1 - \alpha_2 + \alpha_3) \\ & + \text{terms in } I,\ \sigma_3, \text{ and } \sigma_2. \end{aligned} \tag{18}$$

To satisfy (16), first note that

$$\mathrm{Ag}(I) = \mathrm{Ag}(\sigma_3) \equiv 0. \tag{19}$$

This leaves the coefficients of σ_1 and σ_2 in (18) to be separately equated to zero. [It will be seen from (2) of Young[2] that σ_2 terms are generated in (17).] The coefficient of σ_1 is immediately evident from (18). For it to be zero, it is required that

$$D(\alpha_1 - \alpha_2 + \alpha_3) = 0. \tag{20}$$

[This equation does not follow from (14), since the α's are not independent of frequency, as they would be in a homogeneous transformer.]

It can be shown[1,5] that

$$D\alpha_i = \tfrac{1}{2}(t_i^2 - t_{i-1}^2), \tag{21}$$

where

$$t = \frac{\lambda_g}{\lambda}, \tag{22}$$

λ_g being the guide wavelength. In general, each section will have its own value t_i, which is a function of frequency.

It can also be shown[1] that

$$D\theta_i = -t_i^2\theta_i. \tag{23}$$

From (20) and (21),

$$t_1^2 - t_2^2 = \tfrac{1}{2}(t_0^2 - t_3^2). \tag{24}$$

The output to input impedance ratio is

$$R = \frac{Z_3}{Z_0}, \tag{25}$$

[4] L. Young, "Analysis of a transmission cavity wavemeter," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 436–439; July, 1960.

[5] L. Young, "Design of Microwave Stepped Transformers with Applications to Filters," D.Eng. dissertation, The Johns Hopkins Univ., Baltimore, Md.; April, 1959.

and the α's are therefore related by

$$\alpha_1 + \alpha_2 + \alpha_3 = \tfrac{1}{2} \ln R. \tag{26}$$

Eqs. (14) and (24) are necessary but not sufficient conditions for a maximally flat transformer. The other conditions derive from the coefficient of σ_2 in (18) being zero. This yields

$$Ag[E_1(-\alpha_1 - \alpha_2 + \alpha_3) D\theta_1 + E_1(-\alpha_1 + \alpha_2 + \alpha_3) D\theta_2] = 0, \tag{27}$$

after using (2) and (8) of Young.[2] Therefore, from (23) and (1), at center frequency

$$t_1^2 \sinh(-\alpha_1 - \alpha_2 + \alpha_3) + t_2^2 \sinh(-\alpha_1 + \alpha_2 + \alpha_3) = 0. \tag{28}$$

Using (14), this reduces to

$$\frac{\sinh 2\alpha_1}{\sinh 2\alpha_3} = \left(\frac{t_2}{t_1}\right)^2. \tag{29}$$

Eqs. (14), (24) and (29), together with the identity (26), are the complete solution of the maximally flat two-section inhomogeneous transformer problem. When the input and output waveguides are specified, t_0, t_3, and R are known. It is easy to see that there always exists a solution for t_1, t_2, α_1, α_2, and α_3. Moreover, this solution is not unique, since there is only one constraining equation (24) on the two parameters t_1 and t_2.

Each solution will exhibit "maximally flat" behavior in the sense that the reflection coefficient against frequency curve will have a double zero at center frequency. However, some solutions will be flatter than others, and if the one-section transformer may be taken for a guide,[1] we should sometimes expect one solution to be the "flattest maximally flat." We might expect significant differences only as cutoff is approached, or for large values of R, or both.

Instead of finding the optimum solution, as was done for the single-section transformer, this extra one degree of freedom may serve another useful purpose. In rectangular waveguide, the smaller the H-plane steps, the more nearly ideal is the transformer. The parameters t_1 and t_2 may therefore be selected so as to make none of the individual H-plane steps too large. This will reduce first order corrections which might be necessary for nonideal junctions.[1]

Eqs. (14) and (29) may be expressed in terms of the characteristic impedances Z_0, Z_1, Z_2, and Z_3, instead of the log ratios α_1, α_2 and α_3. They become

$$\left(\frac{Z_2}{Z_1}\right)^2 = R, \tag{30}$$

and

$$\left(\frac{Z_1}{Z_0}\right)^2 = \frac{t_1^2 + t_2^2 R^{1/2}}{t_1^2 + t_2^2 R^{-1/2}}, \tag{31}$$

respectively, where R is given by (25).

Numerical Results for Maximally Flat Transformers

To test the theory, several two-section transformers designed by these equations were analyzed. In each case, a maximally flat response, in the sense of a double zero of the reflection coefficient at center frequency, was obtained.

Let a, b again denote guide width and height, respectively (Fig. 1). The suffix numbering is as shown in Fig. 2.

Example 1

Design wavelength,

$\lambda_0 = 9.1$ inches,

$a_0 = 8$ inches, $b_0 = 2$ inches

$a_3 = 5$ inches, $b_3 = 3$ inches.

Selecting either a_1 or a_2 determines the design uniquely. A plot of VSWR against wavelength of five cases is shown in Fig. 3. For example, the dimensions of the transformer with the flattest curve in Fig. 3, are

$a_1 = 8.000$ inches, $b_1 = 3.462$ inches

$a_2 = 5.341$ inches, $b_2 = 3.210$ inches.

There is little change in performance when a_1 is reduced from 8 inches to about 6 or 6.5 inches (Fig. 3). The choice of a_1 will then depend more on practical considerations concerning the departure of the H-plane steps from ideal transformers. The extent to which H-plane steps in rectangular waveguide depart from ideal transformers has already been discussed elsewhere.[1] Unfortunately, each of the transformers in Fig. 3 involves at least one junction which is too far from ideal for a physical model to be expected to follow the ideal transformer theory. An alternative approach would be to use two intermediate transformers of two sections each, and spaced a quarter wave apart, which would reduce the steps to where each transformer is essentially ideal.

Example 2

Design wavelength,

$\lambda_0 = 1.390$ inches

$a_0 = 0.900$ inch, $b_0 = 0.400$ inch,

$a_1 = 0.850$ inch, $b_1 = 0.429$ inch,

$a_2 = 0.771$ inch, $b_2 = 0.417$ inch,

$a_3 = 0.750$ inch, $b_3 = 0.400$ inch,

which conform to the design equations. The VSWR against wavelength response is shown in Fig. 4.

Numerical Results on Broad-Banding

Usually one is concerned with obtaining the best possible performance over a prescribed frequency band, rather than maximally flat response. For the homogeneous transformer, this problem has been solved analytically,[6,7] and the author has made up numerical tables[8] for this case.

Since no exact solution has been found for the inhomogeneous transformer, it seemed worthwhile to try to modify the exact maximally flat solution, using the homogeneous transformer tables as a guide. This attempt turned out to be very successful.

The central idea was to find from the homogeneous transformer tables how much a homogeneous maximally flat transformer had to be modified to give the required bandwidth. This was treated as an additive "correction" to the log ratios, α, or as a multiplicative correction to the impedances, Z. Returning to the inhomogeneous transformer, each guide width a was kept constant, and the "correction" to each Z was absorbed in the height b of the guide.

Example 3

Modifying the transformer with the flattest response curve in Fig. 3, Example 1, for a 30 per cent bandwidth (in reciprocal guide wavelength) yields, with a_1 and a_2 kept the same,

$$a_0 = 8.000 \text{ inches}, \quad b_0 = 2.000 \text{ inches},$$
$$a_1 = 8.000 \text{ inches}, \quad b_1 = 3.512 \text{ inches},$$
$$a_2 = 5.341 \text{ inches}, \quad b_2 = 3.167 \text{ inches},$$
$$a_3 = 5.000 \text{ inches}, \quad b_3 = 3.000 \text{ inches}.$$

The VSWR against wavelength response is shown in Fig. 5. The parent maximally flat case (lowest curve in Fig. 3) is reproduced for comparison (broken line).

In this case $R = 4.761$ and from the tables,[8] a 30 per cent bandwidth transformer yields a maximum VSWR of 1.05. This agrees with the computed value for this inhomogeneous transformer (1.051 in Fig. 5). It is harder to predict the frequency bandwidth, because the guides are not uniformly dispersive. Now $(d\lambda_g/\lambda_g)/(d\lambda/\lambda) = (\lambda_g/\lambda)^2$. For the 8-inch guide, this quantity is 1.48, and for the 5-inch guide 5.88, both at $\lambda_0 = 9.1$ inches. The arithmetic mean is 3.68. We might therefore expect

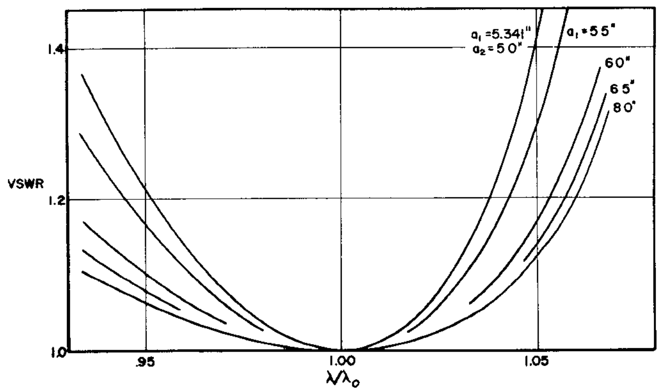

Fig. 3—VSWR vs wavelength of several two-section maximally flat transformers, all from 8×2 inches to 5×3 inches (Example 1).

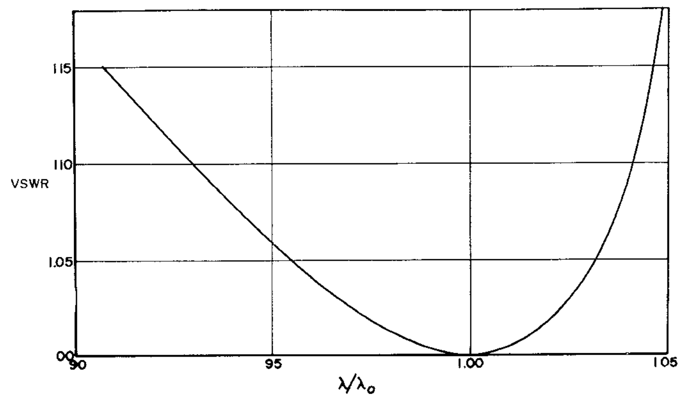

Fig. 4—VSWR vs wavelength of two-section maximally flat transformer 0.9×0.4 inch to 0.75×0.4 inch (Example 2).

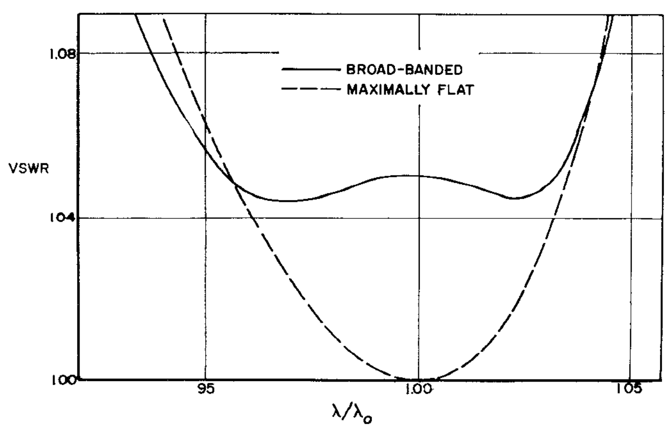

Fig. 5—VSWR vs wavelength of broad-banded and maximally flat transformers (Example 3).

[6] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-5, pp. 36–43; January, 1957.
[7] S. B. Cohn, "Optimum design of stepped transmission-line transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 16–21; April, 1955.
[8] Leo Young, "Tables for cascaded homogeneous quarter-wave transformers," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 233–237, April, 1959; and vol. MTT-8, pp. 243–244, March, 1960.

a bandwidth in the order of $30/3.68 \doteq 8$ per cent. The computed bandwidth for maximum VSWR = 1.051 (Fig. 5) is in very close agreement, being 7.8 per cent.

Example 4

This is Example 2 modified as described above for a 62 per cent bandwidth (in reciprocal guide wavelength). The a's were again kept the same and the new dimensions are:

$$a_0 = 0.900 \text{ inch}, \qquad b_0 = 0.400 \text{ inch},$$
$$a_1 = 0.850 \text{ inch}, \qquad b_1 = 0.437 \text{ inch},$$
$$a_2 = 0.771 \text{ inch}, \qquad b_2 = 0.409 \text{ inch},$$
$$a_3 = 0.750 \text{ inch}, \qquad b_3 = 0.400 \text{ inch}.$$

The VSWR against wavelength response is shown in Fig. 6, with the original maximally flat case (Fig. 4) reproduced for comparison (broken line).

In this case $R = 2.027$, and for a bandwidth of 62 per cent, the tables give a maximum VSWR of 1.09 for the homogeneous transformer. Our inhomogeneous transformer (Fig. 6) is in close agreement with its maximum VSWR of 1.084. The arithmetic mean value of $(d\lambda_g/\lambda_g)/(d\lambda/\lambda) = (\lambda_g/\lambda)^2$ is now $(2.47+7.04)/2 = 4.75$, and therefore we might expect a (frequency) bandwidth in the order of $62/4.75 = 13$ per cent. The computed bandwidth for maximum VSWR = 1.084 is given by Fig. 6 as 12 per cent, which again is in excellent agreement.

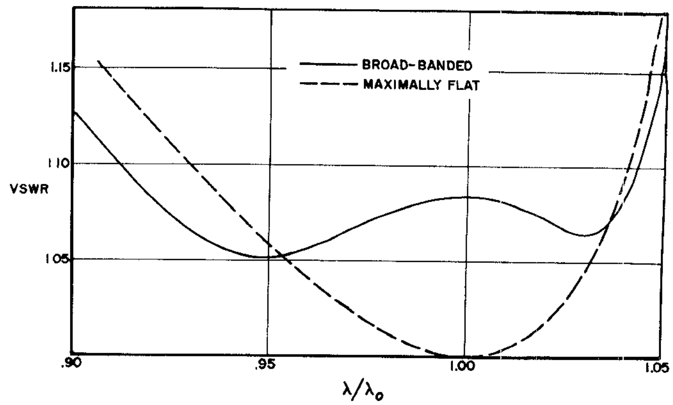

Fig. 6—VSWR vs wavelength of broad-banded and maximally flat transformers (Example 4).

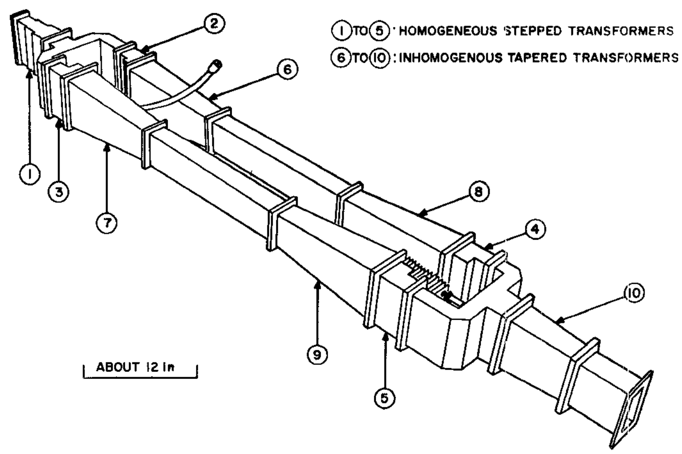

Fig. 7—Waveguide diplexing filter.

Discussion

Inhomogeneous transformers are commonly required when a nonstandard waveguide has to be connected to a component in standard waveguide, or in similar applications calling for small steps. Then a first-order theory is usually adequate. An example of an inhomogeneous transformer of large R and wide band occurred in connection with the design of a diplexing filter,[9] using waveguide sections which were cut off in the lower frequency band and transmitted, but were nearly cut off in the upper band. A transformer had to be built from an 8-inch × 2-inch waveguide to a waveguide of 5 inches × 3 inches, which was approaching cutoff for the upper band of 1250 to 1350 Mc. At that time, no theory was available for inhomogeneous transformers, and a combination E-plane (homogeneous) transformer and a double linear taper were used. A view of the filter is reproduced in Fig. 7. This filter involves no less than ten transformers, five homogeneous and five inhomogeneous. The inhomogeneous ones were all double linear tapers, but could now be designed systematically as quarter-wave transformers, thereby reducing their length and improving their performance. The measured performance[9] of a maximum VSWR of 1.22 in a length of over two feet could be greatly improved with only a two-section ideal transformer (Example 1). As already mentioned, the ideal transformer assumption would not be expected to hold for the large steps involved, but a cascade of two transformers, each of two sections and themselves separated by a quarter-wave section (making a total of five sections), should also give substantially improved performance and would still be shorter than the original design. With the smaller steps involved, it would be possible to make first-order corrections for the transformers not being ideal.

[9] L. Young and J. Q. Owen, "A high power diplexing filter," IRE Trans. on Microwave Theory and Techniques, vol. MTT-7, pp. 384–387; July, 1959.

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